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PRECESSIONAL MOTIONS IN RIGID BODY DYNAMICS AND THE DYNAMICS OF SYSTEMS OF COUPLED RIGID BODIES†

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A brief survey of the results of research on precessional motions in problems of the dynamics of rigid bodies and systems of rigid bodies is given.

Precessional motions of a rigid body may be classed among the most intuitive motions from the mechanical point of view, while at the same time they are widely used in the theory of gyroscopic systems, which is of importance in engineering. As A. Yu. Ishlinskii remarked [1, pp. 353, 354]: "After nutation has damped out, the subsequent slow motion of the axis of the rotor, known as precessional motion, agrees to a significant degree of accuracy with the precessional equations of gyroscope theory ... In gyroscope theory, allowance for the nutational terms in the differential equations of motion of gyroscopic systems turns out to be necessary when studying the behaviour of high-precision gyroscopes".

In the problem of the motion of a heavy rigid body with a fixed point, the regular precession of a Lagrange gyroscope is a classical example of precessional motion. The foundations for the systematic study of precessional motions in rigid body dynamics were laid by Appel'rot [2] and Grioli [3, 4]. Appel'rot considered precession about the vertical in gyroscopes whose inertia ellipsoid was an ellipsoid of rotation, with its centre of gravity lying in the equatorial plane (gyroscopes similar to Kovalevskaya and Goryachev–Chaplygin gyroscopes). He showed that for such gyroscopes, motions in which the constant angle between the principal axis and the vertical is not a rigid angle are dynamically impossible.

Grioli [3, 4] can be credited with numerous results in rigid body dynamics. The most important one is the construction of a new solution of the Euler–Poisson equations describing the regular precession of a heavy rigid body about an inclined axis.

Several publications [5–15]‡ have considered the precessional motion of a rigid body with a fixed point from general positions, proposing methods for investigating the conditions for precession to exist not only in the classical problem but also in its generalizations.

Precessional motions of asymmetric bodies have been investigated in the problem of the motion of a body suspended on a rod, and in the problem of the motion of a system of coupled rigid bodies [10, 11, 13, 16].

1. THE KINEMATIC CONDITIONS FOR PRECESSIONAL MOTIONS OF A RIGID BODY WITH A FIXED POINT

Suppose that, in a fixed space, some fixed direction characterized by a unit vector $\boldsymbol{\nu}$ (for example, the direction of the axis of symmetry of a force field) exists. In addition, assume that $\boldsymbol{\gamma}$ is a unit vector, also unchangeable in space. The origins of the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ coincide with the fixed point O of the rigid body. If we let $\boldsymbol{\omega}$ denote the angular velocity of the body, we have the following equations for $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$

$$\dot{\boldsymbol{\nu}} = \boldsymbol{\nu}\times\boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega} \tag{1.1}$$

where the dot denotes differentiation with respect to time in a moving system of coordinates.

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Let κ_0 be the angle between the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$. Then we have the obvious kinematic relations

$$\mathbf{v}\cdot\mathbf{v} = 1, \quad \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1, \quad \mathbf{v}\cdot\boldsymbol{\gamma} = c_0 \tag{1.2}$$

where $c_0 = \cos\kappa_0$. The motion of the body is said to be precessional if, as long as the body is in motion, the angle between the vectors $\mathbf{a}$ and $\boldsymbol{\gamma}$, where $\mathbf{a}$ is a unit vector rigidly attached to the body ($\dot{\mathbf{a}} = 0$), is constant. This motion may be characterized by an invariant relation

$$\mathbf{a}\cdot\boldsymbol{\gamma}_0 = a_0 \tag{1.3}$$

where $a_0 = \cos\theta_0$ and θ_0 is the constant angle between $\mathbf{a}$ and $\boldsymbol{\gamma}$.

Let us differentiate both sides of Eq. (1.3) along trajectories of the second equation of (1.1). We obtain the equality $\boldsymbol{\omega}\cdot(\mathbf{a}\times\boldsymbol{\gamma}) = 0$, that is

$$\boldsymbol{\omega} = \varphi_1(t)\mathbf{a} + \varphi_2(t)\boldsymbol{\gamma} \tag{1.4}$$

where the case $\mathbf{a}\times\boldsymbol{\gamma} = 0$ is excluded since it leads either to uniform rotation of the body or to pendulum motion. Substituting (1.4) into Eq. (1.1), we obtain

$$\dot{\mathbf{v}} = \varphi_1(t)(\mathbf{v}\times\mathbf{a}) + \varphi_2(t)(\mathbf{v}\times\boldsymbol{\gamma}), \quad \dot{\boldsymbol{\gamma}} = \varphi_1(t)(\boldsymbol{\gamma}\times\boldsymbol{\alpha}) \tag{1.5}$$

As a rule [6–8], the moving system of coordinates is chosen in such a way that $\mathbf{a} = (0, 0, 1)$. Then the relations (1.3) and $\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1$ are satisfied by putting

$$\gamma_1 = a_0'\sin\varphi, \quad \gamma_2 = a_0'\cos\varphi, \quad \gamma_3 = a_0 \tag{1.6}$$

where $a_0' = \sqrt{1-a_0^2} = \sin\theta_0$. Substituting expressions (1.6) into the scalar equations following from the second equation of system (1.5), we obtain $\varphi_1(t) = \dot{\varphi}(t)$. Bearing the first and third equalities of (1.2) and the first equation of (1.5) in mind, we obtain the following representation of the vector $\boldsymbol{\nu}$

$$\mathbf{v} = (c_0 + a_0 b_0'\sin\phi)\boldsymbol{\gamma} - b_0'\mathbf{a}\sin\phi - b_0'(\boldsymbol{\gamma}\times\mathbf{a})\cos\phi \tag{1.7}$$

where

$$b_0' = b_0/a_0', \quad b_0 = \sqrt{1-c_0^2} = \sin\kappa_0, \quad \varphi_2(t) = \dot{\phi}$$

In the approach taken here, θ_0, φ and ϕ are the Euler angles, so defined that the system of coordinates associated with the vector $\mathbf{a}$ is the moving system and that associated with the vector $\boldsymbol{\gamma}$ is the fixed system. Since $\varphi_1(t) = \dot{\varphi}$, $\varphi_2(t) = \dot{\phi}$, we can rewrite formula (1.4) as

$$\boldsymbol{\omega} = \dot{\varphi}\mathbf{a} + \dot{\phi}\boldsymbol{\gamma} \tag{1.8}$$

Consequently, in precessional motions the components of the vector $\boldsymbol{\gamma}$ are expressed in terms of one variable φ, those of the vector $\boldsymbol{\nu}$ in terms of two variables φ and ϕ, and the angular velocity vector has the form (1.8).

A precessional motion is called regular precession if $\dot{\varphi}$ and $\dot{\phi}$ are constants; if one of these functions is a constant, the motion is called semi-regular precession. If neither $\dot{\varphi}$ nor $\dot{\phi}$ is a constant, the motion is called a precession of the general type [3–6].

Besides relations (1.3) and (1.8), there is an equation due to Grioli [3] that can be taken as the condition for motion to be precessional. It follows from the second equation of system (1.1), by the condition $\gamma_3 = a_0$, that $\omega_2\gamma_1 - \omega_1\gamma_2 = 0$, where ω_1 and ω_2 are the first two components of the vector ω in the moving system of coordinates. Differentiation of this equality along trajectories of the kinematic equations for γ_1 and γ_2 yields the equation

$$\dot{\omega}_2\gamma_1 - \dot{\omega}_1\gamma_2 - a_0(\omega_1^2 + \omega_2^2) + \omega_3(\omega_1\gamma_1 + \omega_2\gamma_2) = 0$$

Eliminating γ_1 and γ_2 from this equation by using the equalities $\gamma_1^2 + \gamma_2^2 = a_0'^2$ and $\omega_2\gamma_1 - \omega_1\gamma_2 = 0$, we arrive at Grioli's equation

$$\omega_1\dot{\omega}_2 - \omega_2\dot{\omega}_1 + \omega_3(\omega_1^2 + \omega_2^2) - (\omega_1^2 + \omega_2^2)^{3/2}\operatorname{ctg}\theta_0 = 0 \tag{1.9}$$

Equation (1.9) has only been used in [17]; it has not been widely employed otherwise.

If the kinematic conditions for the motion to be precessional are considered in the form (1.3), (1.6), the components of the vector $\boldsymbol{\nu}$ (1.7) are:

$$\begin{aligned} \nu_1 &= a_0' c_0 \sin\varphi - b_0(\cos\varphi\cos\phi - a_0\sin\varphi\sin\phi) \\ \nu_2 &= a_0'\cos\varphi + b_0(\sin\varphi\cos\phi + a_0\cos\varphi\sin\phi) \\ \nu_3 &= a_0 c_0 - b_0 a_0'\sin\phi \end{aligned} \tag{1.10}$$

Another way of introducing the Euler angles, not including the angle between the vectors **a** and $\boldsymbol{\gamma}$ as one of the Euler angles, is sometimes proposed for investigating precessional motions. Let u, υ and w be the Euler angles, where u is the angle between the vectors **a** and $\boldsymbol{\nu}$. Then

$$\omega_1 = \dot{w}\sin u\sin\upsilon + \dot{u}\cos\upsilon,\quad \omega_2 = \dot{w}\sin u\cos\upsilon - \dot{u}\sin\upsilon,\quad \omega_3 = \dot{\upsilon} + \dot{w}\cos u \tag{1.11}$$

After substituting relations (1.11) into Eq. (1.9), we have

$$(\ddot{w}\dot{u} - \ddot{u}\dot{w})\sin u + \dot{w}(2\dot{u} + \dot{w}^2\sin^2 u)\cos u = (\dot{u}^2 + \dot{w}^2\sin^2 u)^{3/2}\operatorname{ctg}\theta_0 \tag{1.12}$$

Note that when $\kappa_0 = 0$ the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ coincide, and consequently the angles u, υ and w are identical with θ_0, φ and ϕ, respectively. Differential equation (1.12) has the obvious solution $u = \theta_0$. It can be shown that its other solution is [18]

$$\cos(w - w_0) = \frac{\cos u\cos\kappa_0 - \cos\theta_0}{\sin\kappa_0\sin u} \tag{1.13}$$

where w_0 and κ_0 are arbitrary constants. Thus, if the Euler angles are introduced in the traditional manner, that is, are not associated with the characteristic precession directions **a** and $\boldsymbol{\gamma}$, but are defined relative to the axis of symmetry of the force field (the vector $\boldsymbol{\nu}$), the condition for the motion to be precessional has the form (1.13).

2. A METHOD FOR INVESTIGATING PRECESSION ABOUT THE VERTICAL IN THE PROBLEM OF THE MOTION OF A GYROSTAT UNDER THE INFLUENCE OF POTENTIAL AND GYROSCOPIC FORCES

Let us consider the problem of the motion of a gyrostat with a fixed point under the influence of potential and gyroscopic forces, which is described by differential equations of the Kirchhoff class [19, 20]

$$A\dot{\boldsymbol{\omega}} = (A\boldsymbol{\omega} + \boldsymbol{\lambda})\times\boldsymbol{\omega} + \boldsymbol{\omega}\times B\mathbf{v} + s\times\mathbf{v} + \mathbf{v}\times C\mathbf{v},\quad \dot{\mathbf{v}} = \mathbf{v}\times\boldsymbol{\omega} \tag{2.1}$$

which have integrals

$$(A\boldsymbol{\omega}\cdot\boldsymbol{\omega}) - 2(s\cdot\mathbf{v}) + (C\mathbf{v}\cdot\mathbf{v}) = 2E,\quad \mathbf{v}\cdot\mathbf{v} = 1,\quad 2(A\boldsymbol{\omega} + \boldsymbol{\lambda})\cdot\mathbf{v} - (B\mathbf{v}\cdot\mathbf{v}) = 2k \tag{2.2}$$

where E and k are arbitrary constants. In Eqs (2.1) and (2.2), $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the gyrostat, $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the unit vector of the axis of symmetry of the force field, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is the gyrostatic moment, $\mathbf{s} = (s_1, s_2, s_3)$ is a vector collinear with the vector of the generalized centre of mass, A is the inertia tensor, and B and C are 3×3 symmetric matrices. In this paper, Eqs (2.1) will be interpreted as the equations of motion of a gyrostat in a force field which is the superposition of an electric, a magnetic and a Newtonian field [20]. Equations (2.1) are classified as Kirchhoff equations in view of the familiar hydrodynamic analogy between the present problem and the problem of the motion of a body in a fluid [19, 20].

We will consider the precessional motions of the gyrostat about the vertical, that is, we put $\boldsymbol{\gamma} = \boldsymbol{\nu}$ in Eqs (1.3), (1.6) and (1.8). Then

$$\mathbf{v} = (a_0'\sin\varphi, a_0'\cos\varphi, a_0),\quad \boldsymbol{\omega} = \dot{\varphi}\mathbf{a} + \dot{\phi}\mathbf{v} \tag{2.3}$$

Substituting the expression for $\boldsymbol{\omega}$ from (2.3) into the integrals (2.2), we obtain

$$\dot{\varphi}(A\mathbf{a}\cdot\mathbf{v})+\dot{\phi}(A\mathbf{v}\cdot\mathbf{v}) = F_1(v_1,v_2,v_3)$$
$$\dot{\varphi}^2(A\mathbf{a}\cdot\mathbf{a})+2\dot{\varphi}\dot{\phi}(A\mathbf{a}\cdot\mathbf{v})+\dot{\phi}^2(A\mathbf{v}\cdot\mathbf{v}) = F_2(v_1,v_2,v_3) \tag{2.4}$$

where

$$F_1(v_1,v_2,v_3) = k-(\boldsymbol{\lambda}\cdot\mathbf{v})+\frac{1}{2}(B\mathbf{v}\cdot\mathbf{v})$$
$$F_2(v_1,v_2,v_3) = 2E+2(\mathbf{s}\cdot\mathbf{v})-(C\mathbf{v}\cdot\mathbf{v}) \tag{2.5}$$

Relations (2.4) imply [6–8] that

$$\dot{\phi} = \frac{F_1(v_1,v_2,v_3)-\dot{\varphi}(A\mathbf{a}\cdot\mathbf{v})}{(A\mathbf{v}\cdot\mathbf{v})},\quad \dot{\varphi}^2 = \frac{(A\mathbf{v}\cdot\mathbf{v})F_2(v_1,v_2,v_3)-F_1^2(v_1,v_2,v_3)}{F_3(v_1,v_2,v_3)} \tag{2.6}$$

where $F_3(v_1,v_2,v_3) = (A\mathbf{a}\cdot\mathbf{a})(A\boldsymbol{\nu}\cdot\boldsymbol{\nu})-(A\mathbf{a}\cdot\boldsymbol{\nu})^2 > 0$ by virtue of the fact that the matrix A is positive definite and the variables v_1, v_2 and v_3 satisfy the kinematic condition $v_1^2+v_2^2+v_3^2 = 1$.

Since Poisson's equation in system (2.1) holds for the equalities (2.3), we turn to the dynamical equation in (2.1). Substituting expressions (2.3) into the first equation of system (2.1) and using the second equation, we obtain a single vector equality. Since the vectors $\mathbf{a}$, $\boldsymbol{\nu}$ and $A\mathbf{a}\times A\boldsymbol{\nu}$ are independent, we consider the three equations obtained by equating to zero the scalar products of these vectors and the vector on the left of the aforementioned equality. It can be shown that the first two equations are linear combinations of Eqs (2.4), while the third is

$$\begin{aligned}&\dot{\varphi}\dot{\phi}\left[2(A\mathbf{a}\cdot\mathbf{a})(A\mathbf{v})^2-F_3(v_1,v_2,v_3)\mathrm{Tr}(A)-2(A\mathbf{a}\cdot\mathbf{v})(A\mathbf{a}\cdot A\mathbf{v})\right]-\\ &-\dot{\varphi}^2[(A\mathbf{a})^2(A\mathbf{a}\cdot\mathbf{v})-(A\mathbf{a}\cdot\mathbf{a})(A\mathbf{a}\cdot A\mathbf{v})]-\dot{\phi}^2[(A\mathbf{v}\cdot\mathbf{v})(A\mathbf{a}\cdot A\mathbf{v})-(A\mathbf{a}\cdot\mathbf{v})(A\mathbf{v})^2]+\\ &+\dot{\varphi}[(A\mathbf{a}\cdot B\mathbf{v})(A\mathbf{a}\cdot\mathbf{v})-(A\mathbf{a}\cdot\mathbf{a})(A\mathbf{v}\cdot B\mathbf{v})+(A\mathbf{a}\cdot\mathbf{a})(A\mathbf{v}\cdot\boldsymbol{\lambda})-(A\mathbf{a}\cdot\boldsymbol{\lambda})(A\mathbf{a}\cdot\mathbf{v})]+\\ &+\dot{\phi}[(A\mathbf{a}\cdot B\mathbf{v})(A\mathbf{v}\cdot\mathbf{v})-(A\mathbf{a}\cdot\mathbf{v})(A\mathbf{v}\cdot B\mathbf{v})+(A\mathbf{a}\cdot\mathbf{v})(A\mathbf{v}\cdot\boldsymbol{\lambda})-(A\mathbf{v}\cdot\mathbf{v})(A\mathbf{a}\cdot\boldsymbol{\lambda})]+\\ &+(A\mathbf{v}\cdot\mathbf{v})((A\mathbf{a}\cdot A\mathbf{c})-(A\mathbf{a}\cdot\mathbf{s}))+(A\mathbf{a}\cdot\mathbf{v})((A\mathbf{v}\cdot\mathbf{s})-(A\mathbf{v}\cdot C\mathbf{v})) = 0\end{aligned} \tag{2.7}$$

Using expressions (2.6), we can eliminate the quantities $\dot{\varphi}$ and $\dot{\phi}$ in Eq. (2.7) and, on the basis of relations (2.3) and (2.5), reduce it to the form

$$\Phi(\varphi, a_0, E, k, A_{ij}, B_{ke}, C_{mn}, \lambda_i, s_j) = 0 \tag{2.8}$$

where A_{ij}, B_{ke} and C_{mn} are the elements of the matrices A, B and C; λ_i and s_j are the components of the vectors $\boldsymbol{\lambda}$ and $\mathbf{s}$. Equation (2.8) will be called the resolvent in the problem of investigating the conditions for precession of the gyrostat about the vertical to exist, since its representation in the form

$$\sum_{k=1}^{n}(a_k\cos k\varphi+b_k\sin k\varphi) = 0 \tag{2.9}$$

where a_k and b_k are functions of the parameters of problem (2.1), (2.2) and of the constants E, k and θ_0, enable one to determine the necessary conditions for the precession to exist: $a_k = 0$, $b_k = 0$ $(k = 1, \ldots, n)$.

Remark. In the case of regular and semi-regular precession about the vertical, Eqs (2.4), (2.5) and (2.7) must be used.

When investigating the conditions for the existence of precession about an inclined axis ($\boldsymbol{\nu}\neq\boldsymbol{\gamma}$) in a gyrostat, formulae (1.6)–(1.8) must be considered together with Eqs (2.1) and integrals (2.2). Using the first two integrals of (2.2), one can determine $\dot{\varphi}$ and $\dot{\phi}$ as functions of the variables φ and ϕ and the parameters of the problem. One can then obtain an analogue of the resolvent (2.8) in which, unlike (2.8), the two variables φ and ϕ occur. Therefore, along with the resolvent, one must also consider its derivative along trajectories of the equations for $\dot{\varphi}$ and $\dot{\phi}$. Proceeding in this way one can also find the second resolvent and then use the two resolvents to derive an equation of the form (2.9).

3. THE PRECESSIONAL MOTIONS OF A HEAVY RIGID BODY

The equations of the classical problem of the motion of a heavy rigid body follow from system (2.1) under the conditions $\boldsymbol{\lambda} = 0$, $B = 0$, $C = 0$

$$A\dot{\boldsymbol{\omega}} = A\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{s} \times \mathbf{v}, \quad \dot{\mathbf{v}} = \mathbf{v} \times \boldsymbol{\omega} \tag{3.1}$$

These equations have first integrals

$$A\boldsymbol{\omega} \cdot \boldsymbol{\omega} - 2(\mathbf{s} \cdot \mathbf{v}) = 2E, \quad \mathbf{v} \cdot \mathbf{v} = 1, \quad A\boldsymbol{\omega} \cdot \mathbf{v} = k \tag{3.2}$$

Regular precession about the vertical. Putting $\dot{\varphi} = n$, $\dot{\psi} = m$ in the second equality of (2.3), where n and m are constants, we obtain

$$\omega_1 = m\nu_1, \quad \omega_2 = m\nu_2, \quad \omega_3 = n + ma_0, \quad \nu_1 = a_0' \sin nt, \quad \nu_2 = a_0' \cos nt, \quad \nu_3 = a_0 \tag{3.3}$$

Applying the method described above for investigating precession to Eqs (3.1) and (3.2), we obtain the conditions

$$A_{ij} = 0 \ (i \neq j), \quad A_{11} = A_{22}, \quad s_1 = s_2 = 0, \quad mnA_{33} - a_0 m^2 (A_{11} - A_{33}) + s_3 = 0 \tag{3.4}$$

Conditions (3.4) show that the body is a Lagrange gyroscope (in the principal system of coordinates $A = B$, $s_3 \neq 0$). The first three equalities of system (3.3) imply an invariant relation $\boldsymbol{\omega} \cdot \mathbf{s} = \text{const}$, that is, the solution (3.3) is a special case of the Lagrange solution. Regular precession of a Lagrange gyroscope describes the motion of a body which is a superposition of two uniform rotations about axes, one of which is fixed in the body and the other, in space.

Semi-regular precession of the first type about the vertical. Suppose that in (2.3) $\dot{\psi} = m$, where m is a constant

$$\omega_1 = m\nu_1, \quad \omega_2 = m\nu_2, \quad \omega_3 = \dot{\varphi} + ma_0, \quad \nu_1 = a_0' \sin\varphi, \quad \nu_2 = a_0' \cos\varphi, \quad \nu_3 = a_0 \tag{3.5}$$

It has been shown [6, 7] that the conditions for a solution (3.5) of Eqs (3.1) to exist are

$$A_{12} = A_{23} = 0, \quad A_{13}^2 = A_{33}(A_{11} - A_{22}), \quad s_1 = s_2 = 0, \quad s_3 = a_0 m^2 A_{22} \tag{3.6}$$
$$\dot{\varphi} = -m(A_{13} a_0' \sin\varphi + A_{33} a_0) A_{33}^{-1}$$

Let us express the conditions in (3.6) for the matrix components A_{ij} and the components of the vector $\mathbf{s}$ in terms of the principal system of coordinates

$$s_2^* = 0, \quad s_1^* \sqrt{A(B - C)} - s_3^* \sqrt{C(A - B)} = 0$$

where A, B and C are the principle moments of inertia, and s_i^* are the components of $\mathbf{s}$ in the principal system of coordinates. Thus, the body is a Hess gyrostat [21]. If conditions (3.6) are satisfied, solutions (3.5) will satisfy the equality $A\boldsymbol{\omega} \cdot \mathbf{s} = 0$; this implies the following

Theorem 1. In the classical problem of the motion of a heavy rigid body, semi-regular precessional motions about the vertical exist only in the special case of the Hess solution [21].

Note that the semi-regular precession (3.5), (3.6) describes the motion of a Hess gyroscope, which is the superposition of uniform rotation about the vertical and non-uniform rotation $\dot{\varphi}$ of (3.6) about a barycentric axis in the body.

Semi-regular precession of the second type about the vertical. If we put $\dot{\varphi} = n$, $\dot{\psi} \neq m$ in the second equality of (2.3), where n and m are constants, we obtain semi-regular precession of the second type. The following theorem holds.

Theorem 2. In the classical problem of the motion of a rigid body with a fixed point, semi-regular precession of the second type about the vertical is dynamically impossible.

Precession of the general type about the vertical. We will now consider precession of the general type, that is, we will assume that neither of the quantities $\dot{\varphi}$ or $\dot{\phi}$ in the expression for the angular velocity in (2.3) is a constant. We will first characterize the pendulum motions of a body with a fixed point in terms of precessional motion. Suppose that in the second equality of (2.3) $\dot{\phi} = 0$, that is, the angular velocity vector does not change direction, not only in the system of coordinates attached to the body but also in fixed space. It follows from results of Mlodzeyevskii [22] that pendulum motion has the following properties: rotation occurs about a horizontal axis which is a principal axis in the body, the body's centre of mass lies in the principal plane perpendicular to the axis of rotation, and the rotation obeys the law $\dot{\varphi} = \sqrt{\mu_0 + \lambda_0 \sin\varphi}$, where λ_0 and μ_0 are constants.

As regards precession about the vertical, the following results have been established [6, 7].

Theorem 3. If a straight line in the body, making a constant angle with the vertical throughout the motion, is a principal axis in the body, then the precession is either regular or is the motion of a physical pendulum.

Theorem 4. A necessary condition for precession of the general type to exist in a heavy rigid body is $k = 0$, where k is the constant of the angular momentum integral.

Theorem 5. In the problem of the motion of a heavy rigid body, when the vector a lies in the principal plane of the inertia ellipsoid for the fixed point, precession of general type about the vertical takes place only in Dokshevich's solution [23].

In the system of coordinates being used in this paper, this solution is

$$\mathbf{v} = (a_0'\sin\varphi, a_0'\cos\varphi, a_0), \quad \boldsymbol{\omega} = \dot{\varphi}\mathbf{a} + \dot{\phi}\mathbf{v}$$
$$\dot{\varphi} = \sqrt{b_2(b_1 + \sin\varphi)}, \quad \dot{\phi} = b_3\dot{\varphi}(b_1 + \sin\varphi)^{-1} \tag{3.7}$$

The parameters of the problem are subject to the conditions

$$s_2 = 0, \quad A_{12} = A_{23} = 0$$
$$4A_{13}^4 + A_{13}^2(A_{11} - A_{22})(A_{11} + 3A_{22} - 4A_{33}) - A_{11}A_{33}(A_{11} - A_{22})^2 = 0$$
$$\operatorname{ctg}^2\theta_0 = \frac{A_{22}A_{13}^2}{A_{33}[A_{13}^2 - A_{33}(A_{11} - A_{22})]}, \quad \frac{s_3}{s_1} = \frac{A_{13}[(A_{11} - A_{22})(A_{22} - 2A_{33}) + 2A_{13}^2]}{(A_{22} - A_{11})[A_{33}(A_{11} - A_{22}) - A_{13}^2]} \tag{3.8}$$
$$b_1 = \frac{a_0[A_{33}(A_{11} - A_{22}) - 2A_{13}^2]}{a_0'A_{13}(A_{22} - A_{11})}, \quad b_2 = \frac{2s_1(A_{11} - A_{22})a_0'}{A_{33}(A_{11} - A_{22}) - A_{13}^2}, \quad b_3 = \frac{A_{13}}{a_0'(A_{22} - A_{11})}$$
$$E = -s_3a_0 - \frac{s_1a_0A_{33}}{A_{13}}$$

Dokshevich precession has an interesting property: the product of the velocities of rotation about the body's own axis and of precession is a constant: $\dot{\varphi}\dot{\phi} = b_2b_3$. The conditions imposed on the mass distribution in the body according to (3.8), expressed in the principal system of coordinates, show that the body is a Hess gyroscope. This statement is not trivial, since it requires substantial computations [6]. The proof that formula (3.7) indeed describes Dokshevich's solution is based on expressing solution (3.7) in terms of the components of the angular momentum vector in a special system of coordinates and reducing it to Dokshevich's original form [23].

In the general case, the problem of investigating the conditions for precession of the general type to exist remains unsolved.

Regular precession about an inclined axis. Put $\dot{\varphi} = n$, $\dot{\phi} = m$ in Eq. (1.8), where n and m are constants. Then

$$\varphi = nt + \varphi_0, \quad \phi = mt + \phi_0, \quad \omega_1 = m\gamma_1, \quad \omega_2 = m\gamma_2, \quad \omega_3 = n + ma_0$$

The components γ_1 and γ_2 are defined by (1.6). The time-dependence of the vector components ν_i is given by fomulae (1.10).

Substitution of ω_i and γ_i into Eqs (3.1) yields the following conditions for the parameters of the problem [8]

$$n = m, \quad A_{12} = A_{23} = 0, \quad A_{11} = A_{22}, \quad s_1 = s_2 = 0$$
$$\operatorname{tg}\kappa_0 = A_{13}/A_{33}, \quad \theta_0 = \pi/2, \quad s_3^2 = m^4(A_{13}^2 + A_{33}^2) \tag{3.9}$$

Regular precession is described by the solution

$$\omega_1 = m\sin\varphi, \quad \omega_2 = m\cos\varphi, \quad \omega_3 = m, \quad \varphi = mt + \varphi_0$$
$$\nu_1 = \cos\kappa_0\sin\varphi - \sin\kappa_0\cos^2\varphi, \quad \nu_2 = \cos\kappa_0\cos\varphi + \sin\kappa_0\sin\varphi\cos\varphi \tag{3.10}$$
$$\nu_3 = -\sin\kappa_0\sin\varphi$$

Under these conditions the components of the vector $\boldsymbol{\gamma}$ are $\gamma_1 = \sin\varphi$, $\gamma_2 = \cos\varphi$, $\gamma_3 = 0$. The conditions that must be satisfied by the moments of inertia and the components of the vector **s** of (3.9) may be written as

$$s_2^* = 0, \quad s_1^*\sqrt{C-B} - s_3^*\sqrt{B-A} = 0$$

where A, B and C are the principal moments of inertia, and s_i^* are the components of the vector **s** in the principal systems of coordinates, that is, a rigid body performing regular precession is a Grioli gyroscope [4].

Regular Grioli precession [4] describes motion which is the superposition of two uniform rotations at equal velocities about a barycentric axis in the body and about an axis orthogonal to it in space.

Precession about a horizontal axis [17]. Precession about a horizontal axis is characterized by the following properties

$$\mathbf{a} = \mathbf{s}/s, \quad \mathbf{s}\cdot\boldsymbol{\gamma} = 0, \quad \boldsymbol{\nu}\cdot\boldsymbol{\gamma} = 0$$

that is, the following substitutions should be made in formulae (1.6)–(1.8)

$$\theta_0 = \pi/2, \quad \kappa_0 = \pi/2, \quad \gamma_1 = \sin\varphi, \quad \gamma_2 = \cos\varphi, \quad \gamma_3 = 0$$
$$\nu_1 = -\cos\varphi\cos\phi, \quad \nu_2 = \sin\varphi\cos\phi, \quad \nu_3 = -\sin\phi \tag{3.11}$$
$$\omega_1 = \dot{\phi}\gamma_1, \quad \omega_2 = \dot{\phi}\gamma_2, \quad \omega_3 = \dot{\varphi}$$

Bressan [17], studying the precession (3.1) in Hess's solution [21], used Grioli's equation (1.9). In the system of coordinates being used here, the condition for a rigid body to be a Hess gyroscope is determined by the first three equalities of (3.6). Substituting formulae (3.11) into Eqs (3.1) and integrals (3.2), we obtain

$$\dot{\varphi} = -A_{13}/A_{33}\dot{\phi}\sin\varphi, \quad \dot{\phi} = \sqrt{2(E - s_3\sin\phi)A_{22}^{-1}} \tag{3.12}$$

It follows from the second equation of system (3.12) that $\phi = \phi(t)$ is an elliptic function of time. The function $\varphi(t)$ may be found from the first equation of the system.

For the moment, no other types of precessional motion have been found in the classical problem (3.1). Analysis of the conditions imposed on the mass distribution of a rigid body in the classes of precessional motion of a heavy rigid body described above shows that the only bodies that precess in a uniform force field are Lagrange gyroscopes (dynamically symmetric bodies with centre of mass on an axis of symmetry), Hess gyroscopes (bodies whose centres of mass lie on a perpendicular to a circular section of the gyration ellipsoid) and Grioli gyroscopes (bodies whose centres of mass lie on a perpendicular to a circular section of the inertia ellipsoid). As a corollary of Theorem 3, we find that gyroscopes of the Kovalevskaya and Goryachev–Chaplygin types admit of only trivial precession – rotation about a horizontal axis in space.

4. PRECESSIONAL MOTION OF A ZHUKOVSKII GYROSTAT [24]

Let us consider Zhukovskii's solution [24]. Putting $\mathbf{s} = \mathbf{0}, B = 0, C = 0$ in Eqs (2.1) and (2.2), we obtain

$$A\dot{\boldsymbol{\omega}} = (A\boldsymbol{\omega}+\boldsymbol{\lambda})\times\boldsymbol{\omega}, \quad \dot{\mathbf{v}} = \mathbf{v}\times\boldsymbol{\omega}, \quad A\boldsymbol{\omega}\cdot\boldsymbol{\omega} = 2E \quad (A\boldsymbol{\omega}+\boldsymbol{\lambda})\cdot\mathbf{v} = k \tag{4.1}$$

It follows from the first equation of system (4.1) that the vector $A\boldsymbol{\omega} + \boldsymbol{\lambda}$ is fixed in space. It is natural to take the vector $\boldsymbol{\nu}$ in the form $\mathbf{v} = (A\boldsymbol{\omega} + \boldsymbol{\lambda})/x_0$, where $x_0 = |A\boldsymbol{\omega} + \boldsymbol{\lambda}|$. Poisson's equation in (4.1) holds by virtue of the first equation in system (4.1). The dynamical equations of (4.1) have two first integrals

$$A_1\omega_1^2+A_2\omega_2^2+A_3\omega_3^2 = h^2 \quad (A_1\omega_1+\lambda_1)^2+(A_2\omega_2+\lambda_2)^2+(A_3\omega_3+\lambda_3)^2 = x_0^2 \tag{4.2}$$

It has been shown [25] that if the conditions

$$\lambda_2 = 0 \quad (A_2h^2-x_0^2)(A_2-A_1)(A_3-A_2) = A_2[\lambda_3^2(A_2-A_1)-\lambda_1^2(A_3-A_2)] \tag{4.3}$$

are satisfied, the Zhukovskii gyrostat will precess semi-regularly about the vector $\boldsymbol{\nu}$. Under these conditions the components of the vector $\mathbf{a}$ satisfy the equalities

$$a_2 = 0, \quad a_1\sqrt{A_1(A_3-A_2)}-a_3\sqrt{A_3(A_1-A_2)} = 0$$

which are analogous to the equalities relating the components of the vector of the centre of mass and the principal moments of inertia for a Hess gyrostat. This means that, if conditions (4.3) are satisfied, a straight line in the gyrostat orthogonal to a circular section of the gyration ellipsoid centred at the fixed point will make a constant angle with the vector $\boldsymbol{\nu}$. This angle is defined by

$$\cos\theta_0 = \frac{\sqrt{A_2}(\lambda_1\sqrt{A_3(A_3-A_2)}-\lambda_3\sqrt{A_1(A_2-A_1)})}{x_0\sqrt{(A_3-A_2)(A_2-A_1)(A_3-A_1)}}$$

Under conditions (4.3), the curve in which the surfaces (4.2) intersect is in the plane

$$\sqrt{A_1(A_3-A_2)}[\omega_1(A_2-A_1)-\lambda_1]+\sqrt{A_3(A_2-A_1)}[\omega_3(A_3-A_2)+\lambda_3] = 0 \tag{4.4}$$

Working from the first equality of system (4.2), Eq. (4.4) and Eqs (4.1), we obtain the relations

$$\omega_1 = \alpha_0\sin\varphi+\beta_0, \quad \omega_2 = \gamma_0\cos\varphi, \quad \omega_3 = \varepsilon_0\sin\varphi+\sigma_0, \quad \dot{\varphi} = d_0+d_1\sin\varphi \tag{4.5}$$

in which all the coefficients depend on the parameters of the problem. Transforming to a system of coordinates associated with the vector $\mathbf{a}$, one can show that the solution (4.5) describes semi-regular precession of the first type of a Zhukovskii gyrostat about the vector $A\boldsymbol{\omega} + \boldsymbol{\lambda}$. For such precession, as in the case of (3.6), the function $\varphi(t)$ satisfies an equation of the form $\dot{\varphi} = \mu_0 + \mu_1\sin\varphi$, where μ_0 and μ_1 are constants.

5. EXAMPLES OF PRECESSION ABOUT THE VERTICAL IN PROBLEM (2.1)

Regular precession. Put $\dot{\varphi} = n$, $\dot{\psi} = m$ in formulae (2.3), (2.6) and (2.7), where n and m are constants. This yields the following conditions for the parameters of the problem [12]

$$\begin{aligned}
&A_{12} = 0, \quad B_{12} = 0, \quad C_{12} = 0, \quad 2m(A_{22}-A_{11})+B_{11}-B_{22} = 0\\
&m^2(A_{22}-A_{11})+C_{22}-C_{11} = 0, \quad m^2A_{13}-mB_{13}-C_{13} = 0\\
&m^2A_{23}-mB_{23}-C_{23} = 0\\
&s_1 = a_0C_{13}+mA_{13}(ma_0+n), \quad s_2 = a_0C_{23}+mA_{23}(ma_0+n)\\
&\lambda_1 = B_{13}a_0-A_{13}(2ma_0+n), \quad \lambda_2 = B_{23}a_0-A_{23}(2ma_0+n)\\
&mn(A_{22}+A_{33}-A_{11})+m\lambda_3+B_{11}(n+ma_0)-\\
&-B_{33}ma_0-a_0m^2(A_{11}-A_{33})+a_0(C_{11}-C_{33})+s_3 = 0
\end{aligned} \tag{5.1}$$

In regular precession, the functions $\omega_i(t)\nu_i(t)$ are defined by formulae (3.3). If $B_{ij} = 0$ $(i, j = 1, 2, 3)$, conditions (5.1) imply the conditions outlined in [26].

Let us consider the case $m = 0$ ($\boldsymbol{\omega}$ = na), in which the gyrostat rotates uniformly about a fixed axis in space other than the vertical. System (5.1) implies the conditions

$$\begin{gathered} A_{12} = 0, \quad B_{12} = 0, \quad C_{12} = C_{13} = C_{23} = 0, \quad B_{11} = B_{22}, \quad C_{11} = C_{22}, \quad s_1 = s_2 = 0 \\ \lambda_1 = B_{13}a_0 - nA_{13}, \quad \lambda_2 = B_{23}a_0 - nA_{23}, \quad nB_{11} + s_3 + a_0(C_{11} - C_{33}) = 0 \end{gathered} \tag{5.2}$$

and when these conditions are satisfied, the gyrostat will rotate uniformly about an inclined axis $a_0' \neq 0$. This case cannot occur in the classical problem, since if $B_{ij} = 0$, $C_{kl} = 0$ $(i, j = 1, 2, 3;\ k, l = 1, 2, 3)$, it follows from (5.2) that $s_3 = 0$ (the centre of mass of the gyrostat is fixed).

Semi-regular precession of the first type ($\dot{\phi} = m$, $\dot{\varphi} \neq n$). We will present an example of precession of the first type in the case when

$$\dot{\varphi} = m(p_0 + q_0 \sin\varphi) \tag{5.3}$$

where $p_0^2 = 1 + q_0^2$. This example is interesting because the motion of the gyroscope is not only precessional (see (2.3)) but also isoconical [9], with the moving and fixed hodographs of the angular velocity of the gyrostat symmetrical about a plane tangent to them. The method outlined in Section 2 yields the following conditions for the parameters of the problem

$$\begin{aligned} & A_{23} = 0, \quad B_{12} = mA_{12}, \quad C_{12} = -m^2 A_{12}, \quad C_{23} = -mB_{23} \\ & \lambda_2 = a_0 B_{23}, \quad s_2 = -a_0 m B_{23}, \quad 2mq_0 A_{13} = a_0'[2m(A_{22} - A_{11}) + B_{11} - B_{22}] \\ & m^2 q_0^2 A_{33} = a_0'^2 \left[m^2(A_{11} - A_{22}) - m(B_{11} - B_{22}) + C_{22} - C_{11}\right] \\ & a_0'(s_1 + m\lambda_1) = m^2 p_0 q_0 A_{33} + a_0 a_0'(C_{13} + B_{13}m - A_{13}m^2) \\ & mq_0(B_{11} + B_{22} + 2mA_{33}) = 2a_0'(C_{13} + B_{13}m - A_{13}m^2) \\ & mp_0(B_{11} + B_{22} + 2mA_{33}) + 2a_0\left[C_{22} - C_{33} + m(B_{22} - B_{33}) - m^2(A_{22} - A_{33})\right] + 2(s_3 + \lambda_3) = 0 \end{aligned} \tag{5.4}$$

To simplify the notation, the parameters p_0 and q_0 have not been eliminated in Eqs (5.4).

For this type of precession, the main variables of the problem are given by formulae (3.5) in which $\varphi(t)$ has the form

$$\varphi(t) = 2\,\mathrm{arctg}\left[p_0\left(1 - q_0 \,\mathrm{tg}\frac{mt}{2}\right)^{-1} \mathrm{tg}\frac{mt}{2}\right] \tag{5.5}$$

Semi-regular precession of the second type. Following the approach described in [15], we put

$$\varphi = nt, \quad \dot{\phi} = \frac{n}{d_0 + e_0 \sin\varphi}; \quad d_0^2 = 1 + e_0^2 \tag{5.6}$$

in the second equality of (2.3), that is, we again assume that the motion of the gyrostat is precessional and isoconical. It follows from (5.6) that $\phi(t)$ is an elementary function of time:

$$\phi(t) = 2\,\mathrm{arctg}\left[\left(d_0 + e_0 \,\mathrm{tg}\frac{nt}{2}\right)^{-1} \mathrm{tg}\frac{nt}{2}\right]$$

The conditions for precession (5.6) to exist are [15]

$$\begin{aligned} & A_{12} = A_{23} = 0, \quad B_{12} = B_{23} = 0, \quad C_{12} = C_{13} = C_{23} = 0, \quad B_{11} = B_{22} \\ & C_{11} = C_{22}, \quad s_1 = s_2 = 0, \quad \lambda_2 = 0, \quad \lambda_1 = a_0 B_{13} + n e_0^{-1} a_0'(A_{22} - A_{11}) \\ & s_3 = a_0(C_{33} - C_{11}) + n e_0^{-1}(a_0' B_{13} - c_0 B_{11}) \end{aligned}$$

$$\lambda_3 a_0'^2 = a_0 a_0'^2 (B_{33} - B_{11}) + e_0 d_0^{-1} a_0 A_{13} - a_0'(A_{11} - A_{22} + A_{33}) +$$

$$+ d_0^{-1} a_0 a_0'(A_{22} - A_{33}) - n^{-1} e_0^{-1} d_0 a_0'^2 B_{13}; \quad d_0^2 = 1/(1-\lambda^2), \quad e_0^2 = \lambda^2/(1-\lambda^2)$$

$$\lambda^2 (a_0'^2 A_{22} + a_0^2 A_{33}) - 2\lambda a_0 a_0' A_{13} - a_0'^2 (A_{22} - A_{11}) = 0 \tag{5.7}$$

$$(A_{22} + \sigma A_{33})(Q_1\sigma + Q_0)^2 - 4\sigma A_{13}^2 (R_1\sigma + R_0)(Q_1\sigma + Q_0) -$$

$$- 4\sigma A_{13}^2 (A_{22} - A_{11})(R_1\sigma + R_0)^2 = 0; \quad \sigma = a_0^2/a_0'^2$$

$$Q_0 = A_{22}[A_{13}^2 - A_{11}(A_{22} - A_{11})]$$

$$Q_1 = A_{13}^2 (A_{22} - A_{11} + A_{33}) + A_{33}(A_{22} - A_{11})(A_{33} - A_{11} - A_{22})$$

$$R_0 = A_{22}(A_{11} - A_{22} + A_{33}), \quad R_1 = A_{11}A_{33} - A_{13}^2$$

It has been shown [15] that conditions (5.7) are solvable.

Precession of the general type. Examples of precession of the general type were indicated in [14, 15]. Dokshevich's solution (3.7), (3.8) has been generalized [14]. We note that under the conditions for existence found above, Eqs (3.8) are also satisfied except for the restriction on s_3/s_1 and the last relation. The other conditions are

$$C_{ij} = 0, \quad (i \neq j), \quad B_{12} = B_{23} = 0, \quad B_{11} = B_{22}, \quad C_{11} = C_{22}, \quad s_0 = 0$$

$$\lambda_2 = 0, \quad \lambda_1 = a_0 B_{13}, \quad \lambda_3 = -b_1 b_3^{-1} B_{11} + a_0 (B_{33} - B_{11}), \quad B_{11} = a_0' b_3 B_{13}$$

$$s_3 = a_0 (C_{33} - C_{11}) + \frac{s_1 A_{13} \lfloor (A_{11} - A_{22})(A_{22} - 2A_{33}) + 2A_{13}^2 \rfloor}{(A_{22} - A_{11})[A_{33}(A_{11} - A_{22}) - A_{13}^2]}$$

The following formulae define precession of the general type [15]

$$\boldsymbol{\omega} = \dot{\varphi}(\mathbf{a} + \mathbf{v}), \quad \dot{\varphi}^2 = \mu_1 + \mu_2 \sin\varphi, \quad \mathbf{v} = (a_0' \sin\varphi, a_0' \cos\varphi, a_0)$$

It takes place under the following conditions

$$A_{ij} = 0 \; (i \neq j), \quad B_{ij} = 0 \; (i \neq j), \quad B_{11} = B_{22}, \quad C_{12} = C_{23} = 0, \quad C_{11} = C_{22}$$

$$s_2 = 0, \quad \lambda_1 = \lambda_2 = 0, \quad a_0 = \frac{A_{11}}{A_{11} - A_{33}}, \quad s_1 = \frac{C_{13}(A_{11}^2 + A_{33}^2 - 4A_{11}A_{33})}{(A_{33} - A_{11})(2A_{33} - A_{11})}$$

$$\lambda_3 = \frac{A_{11}B_{33} + B_{11}(A_{33} - 2A_{11})}{A_{11} - A_{33}}, \quad \mu_1 = \frac{s_1(A_{11} - A_{33}) + A_{11}(C_{11} - C_{33})}{(A_{33} - A_{11})^2}$$

$$\mu_2 = \frac{2A_{11}C_{13}}{(A_{33} - A_{11})(A_{11} - 2A_{33})}$$

6. THE PRECESSION OF A RIGID BODY SUSPENDED ON A ROD

The equations of motion of a body suspended on a rod may be written as follows [27]:

$$A\dot{\boldsymbol{\omega}} + m\rho^2[(\mathbf{e} \cdot \dot{\boldsymbol{\omega}})\mathbf{e} - \boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \mathbf{e})(\boldsymbol{\omega} \times \mathbf{e})] = A\boldsymbol{\omega} \times \boldsymbol{\omega} + R(t)\rho(\mathbf{e} \times \mathbf{r}) \tag{6.1}$$

$$mr_0[\ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + (\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r}] = m\rho[\mathbf{e} \times \dot{\boldsymbol{\omega}} - (\boldsymbol{\omega} \cdot \mathbf{e})\boldsymbol{\omega} + \omega^2\mathbf{e}] + P\mathbf{v} - R(t)\mathbf{r} \tag{6.2}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of a body of mass m, $\boldsymbol{v}$ is a unit vector in the direction of the force of gravity $\mathbf{P} = mg\boldsymbol{\nu}$, $\mathbf{r}$ is a unit vector pointing along a rod of length r_0 from a fixed point Q_1 to the point O of suspension of the body, $R(t)$ is the magnitude of the reaction, $\mathbf{e}$ is a unit vector directed from the point O to the centre of mass C, A is the inertia tensor of the body relative to the point O, $\rho = OC$, and g is the acceleration due to gravity.

Suppose the mass distribution of the body at the point of suspension satisfies Hess's conditions [21]. We choose the moving system of coordinates so that $\mathbf{e} = (0, 0, 1)$. Then

$$A_{12} = A_{23} = 0, \quad A_{13}^2 = A_{33}(A_{11} - A_{22})$$

We multiply both sides of Eq. (6.1) scalarly by the vector $\mathbf{e}$

$$(A\boldsymbol{\omega}\cdot\mathbf{e})^{\cdot} = A\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\times\mathbf{e}). \tag{6.3}$$

By virtue of the conditions imposed on A_{ij} it follows from (6.3) that $A\boldsymbol{\omega}\cdot\mathbf{e} = 0$ is an invariant relation. Let us assume that the angular velocity vector of the body has the form $\boldsymbol{\omega} = \dot{\varphi}\mathbf{e} + \omega_0\boldsymbol{\nu}$, that is, the motion is semi-regular precession of the first type. Obviously, $\mathbf{e}\cdot\boldsymbol{\nu} = a_0$, that is, by the equality $\boldsymbol{\nu}\cdot\boldsymbol{\nu} = 1$ and (6.3), we have

$$\dot{\varphi} = -\omega_0 A_{33}^{-1}(A_{13}a_0'\sin\varphi + A_{33}a_0), \quad \mathbf{v} = (a_0'\sin\varphi, a_0'\cos\varphi, a_0) \tag{6.4}$$

It has been shown [11] that the components of the vector $\mathbf{r}$ are

$$r_1 = \sin\varphi\sin\phi_0, \quad r_2 = \cos\varphi\sin\phi_0, \quad r_3 = \cos\phi_0 \tag{6.5}$$

The conditions relating the parameters of problem (6.1), (6.2)

$$mr_0\omega_0^2a_0'\rho(a_0\sin\phi_0 - a_0'\cos\phi_0)\sin\phi_0 - m\rho^2a_0'\omega_0^2(a_0\cos\phi_0 + a_0'\sin\phi_0) -$$
$$- mg\rho a_0\sin\phi_0 + \omega_0^2a_0a_0'A_{22}\cos\phi_0 = 0$$
$$mr_0\rho\omega_0^2a_0(a_0'\cos\phi_0 - a_0\sin\phi_0) - mg\rho a_0' + \omega_0^2a_0a_0'A_{22} = 0$$

may serve as the conditions for semi-regular precession of a body suspended on a rod to exist. Note that the angles between two vectors of the triple $\mathbf{e}$, $\mathbf{r}$ and $\mathbf{v}$ are constant. The velocity of the point of suspension is also constant

$$v_0 = r_0|\omega_0(a_0'\cos\varphi_0 - a_0\sin\phi_0)|$$

Thus, the existence of semi-regular precession of a body suspended on a rod has been demonstrated [1]. However, that does not mean that one can directly carry over the results of the solution of the classical problem to the solution of problem (6.1), (6.2). For example, it has been proved [10] that in the problem of the motion of a body suspended on a rod regular precessional motions about an inclined axis (Grioli precession) do not exist.

7. THE PRECESSION OF A SYSTEM OF COUPLED RIGID BODIES

Let us consider a system or rigid gyrostats $S_1, S_2, \ldots, S_n$ $(n \geqslant 1)$, connected by hinges, in a uniform force field [24]. Gyrostat S_1 has a fixed point O_1, and gyrostats S_i $(i = 2, \ldots, n)$ are linked together as a "chain" by ideal spherical hinges $O_2, \ldots, O_n$ in such a way that each gyrostat's centre of mass and hinges are collinear [13]. Let O_1xyz be a fixed system of coordinates whose axis O_1z with unit vector $\boldsymbol{\nu}$ points vertically upwards. The moving systems of coordinates $C_ix_1^{(i)}x_2^{(i)}x_3^{(i)}$ are associated with the centres of mass C_i of the gyrostats and directed along the principal axes of the inertia ellipsoid.

Suppose $A^{(i)}$ is the central inertia tensor of the ith gyrostat, with diagonal elements $A_1^{(i)} < A_2^{(i)} < A_3^{(i)}$ and $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}^{(i)}$ are the vectors of absolute angular velocity of the body of the ith gyrostat and its gyrostatic moment. Let $\mathbf{r}^{(i)}$, $\mathbf{r}^{(i+1)}$ $(\mathbf{r}^{(i+1)} = k^{(i)}\mathbf{r}^{(i)})$ be the radius vectors of the points O_i and O_{i+1} relative to the point C_i, and $\omega_j^{(i)}$, $\lambda_j^{(i)}$ and $e_j^{(i)}$ $(j = 1, 2, 3)$ the projections of the vectors $\boldsymbol{\omega}^{(i)}$, $\boldsymbol{\lambda}^{(i)}$ and $\mathbf{r}^{(i)}$ onto the $x_i^{(i)}$ axis. We shall assume that $k^{(n)} = 0$.

The equations of motion of the system are

$$A^{(i)}\dot{\boldsymbol{\omega}}^{(i)} + \boldsymbol{\omega}^{(i)} \times (A^{(i)}\boldsymbol{\omega}^{(i)} + \boldsymbol{\lambda}^{(i)}) = \mathbf{r}^{(i)} \times (\mathbf{R}^{(i)} - k^{(i)}\mathbf{R}^{(i+1)}) \tag{7.1}$$

where $\mathbf{R}^{(i)}$ is the reaction exerted by the body S_{i-1} and $\mathbf{R}^{(i+1)}$ is the reaction exerted by the body S_{i+1}.
In scalar notation Eqs (7.1) give

$$\begin{aligned} &A_1^{(i)}\dot{\omega}_1^{(i)} + (A_3^{(i)} - A_2^{(i)})\omega_2^{(i)}\omega_3^{(i)} + \lambda_3^{(i)}\omega_2^{(i)} - \lambda_2^{(i)}\omega_3^{(i)} = \\ &= e_2^{(i)}(R_3^{(i)} - k^{(i)}R_3^{(i+1)}) - e_3^{(i)}(R_2^{(i)} - k^{(i)}R_2^{(i+1)}) \quad (1\ 2\ 3) \end{aligned} \tag{7.2}$$

Assume that $e_2^{(i)} = 0$, $\lambda_2^{(i)} = 0$ $(i = 1, \ldots, n)$. Then, on the assumption that

$$e_1^{(i)}\sqrt{A_1^{(i)}(A_3^{(i)} - A_2^{(i)})} - e_3^{(i)}\sqrt{A_3^{(i)}(A_2^{(i)} - A_1^{(i)})} = 0 \tag{7.3}$$

Eqs (7.2) imply the invariant relations

$$A_1^{(i)}e_1^{(i)}\omega_1^{(i)} + A_3^{(i)}e_3^{(i)}\omega_3^{(i)} + \lambda_3^{(i)}e_1^{(i)} - \lambda_1^{(i)}e_3^{(i)} = 0 \tag{7.4}$$

Thus, if the centre of mass C_i and the hinges O_i, O_{i+1} of each gyrostat S_i lie on one straight line perpendicular to a circular section of the gyrostat's central ellipsoid of gyration, and the gyrostatic moment $\lambda^{(i)}$ lies in a plane perpendicular to the same circular section, then Eqs (7.2) admit of the system of invariant relations (7.4). Obviously, when $n = 1$, $\lambda_1^{(1)} = \lambda_3^{(1)} = 0$, we obtain the Hess case [12].

Just as in the problem of a body suspended on a rod, the invariant relations (7.4) can be satisfied by considering the class of semi-regular precessional motions of bodies S_i (putting $\lambda^{(i)} = 0$):

$$\boldsymbol{\omega}^{(i)} = \dot{\varphi}_i\mathbf{k}_i + \omega_0\mathbf{v}, \quad \dot{\varphi}_i = \omega_0(a_i + b_i\sin\varphi_i); \quad \mathbf{k}_i = \mathbf{O}_i\mathbf{O}_{i+1}/O_iO_{i+1} \tag{7.5}$$

It can be shown [13] that when the equalities (7.4) hold, the motion of the chain of heavy rigid bodies may be divided into two motions: 1) motion of the polygonal line $O_1O_2 \ldots O_{n+1}$ as a chain of heavy rods of masses equal to those of the corresponding bodies and central moments of inertia $A_2^{(i)}$; 2) rotation of the body about the segments of the polygonal line. But if in addition relations (7.5) are true, then semi-regular precession of the system $S_1, \ldots, S_n$ is the superposition of uniform rotation, at angular velocity ω_0, of the polygonal line $O_1O_2 \ldots O_{n+1}$ as a rigid body about the vertical, and rotation of the bodies S_i obeying relations (7.5). Note that under those condition all the segments lie in a single vertical plane, while some of the b_i's in (7.5) may equal zero.

Precessional motions of a coupled system of two asymmetric heavy bodies have been considered on the assumption that one body is precessing and the other is rotating uniformly about the vertical [16].

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